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STUDIES OF INTERNAL DISPLACEMENTS IN  
SOLID PROPELLANT GRAINS

(Evaluation of Phenomenological Relation  
for Viscoelastic Materials)

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L. U. Rastrelli  
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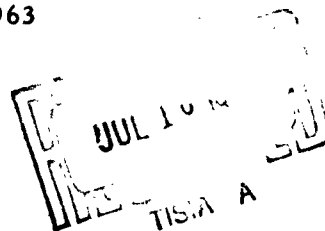
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## I. INTRODUCTION



The purpose of the study presented herein is to demonstrate the applicability of a phenomenological relation between stress, strain rate and temperature in describing the behavior of viscoelastic materials, in general, and solid propellants in particular. The basic relation and its specific equations associated with various testing conditions are discussed in the appendix.

The procedure was to extract from the literature those test data which would not only provide the required material constants, but at the same time permit an evaluation of the relation's range of applicability.

Although the majority of the data uncovered left much to be desired in the way of completeness (from the viewpoint of utilization in this evaluation), nevertheless, it was possible to detect certain encouraging trends. The manner in which the available data were utilized and the interpretive conclusions are presented.



## II. STRESS-STRAIN (CONSTANT STRAIN RATE)

The majority of published experimental information for visco-elastic materials is in the form of stress-strain, ultimate stress, or ultimate strain, constant strain rate data. In order to extract from these types of data the governing material parameters  $\sigma_0$ ,  $\frac{\Delta H}{R}$  and  $s$  (or  $\log_e s$ ) it is necessary to initially assume that Equation (A5), with its inherent limitations regarding the values of  $\beta$  and  $M$ , applies when the strains are "large." Two methods of extracting the constants from the data were derived.

The first method utilizes Equation (A5) in the form

$$\frac{\sigma_l}{\sigma_0} + \log_e s - \frac{\Delta H}{RT} = \log_e \frac{\dot{\epsilon}}{T} \quad (1)$$

For any one test condition, this equation is linear in  $\frac{1}{\sigma_0}$ ,  $\log_e s$  and  $\frac{\Delta H}{R}$ , with constant coefficients. For any three test conditions of a material yielding stress data at high strain, we have a system of three linear equations in three unknowns. The determinant of the system is

$$\Delta = \begin{vmatrix} \sigma_{l1} & 1 & -\frac{1}{T_1} \\ \sigma_{l2} & 1 & -\frac{1}{T_2} \\ \sigma_{l3} & 1 & -\frac{1}{T_3} \end{vmatrix}$$

$$\Delta = \frac{\sigma_{l3} - \sigma_{l2}}{T_1} + \frac{\sigma_{l1} - \sigma_{l3}}{T_2} + \frac{\sigma_{l2} - \sigma_{l1}}{T_3} \quad (2)$$

Data at no less than two temperatures are necessary, since when

$T_1 = T_2 = T_3$ ,  $\Delta = 0$ , (i. e., the system is inconsistent and has no solution).

Solving for  $\frac{1}{\sigma_0}$ ,  $\log_e s$  and  $\frac{\Delta H}{R}$ , we have

$$\Delta \cdot \frac{1}{\sigma_0} = \begin{vmatrix} \log_e \frac{\dot{\epsilon}_1}{T_1} & 1 & -\frac{1}{T_1} \\ \log_e \frac{\dot{\epsilon}_2}{T_2} & 1 & -\frac{1}{T_2} \\ \log_e \frac{\dot{\epsilon}_3}{T_3} & 1 & -\frac{1}{T_3} \end{vmatrix}$$

$$\Delta \cdot \frac{1}{\sigma_0} = \frac{1}{T_1} \left( \log_e \frac{\dot{\epsilon}_3}{T_3} - \log_e \frac{\dot{\epsilon}_2}{T_2} \right) + \frac{1}{T_2} \left( \log_e \frac{\dot{\epsilon}_1}{T_1} - \log_e \frac{\dot{\epsilon}_3}{T_3} \right) + \frac{1}{T_3} \left( \log_e \frac{\dot{\epsilon}_2}{T_2} - \log_e \frac{\dot{\epsilon}_1}{T_1} \right) \quad (3)$$

$$\Delta \cdot \log_e s = \begin{vmatrix} \sigma_{l1} \log_e \frac{\dot{\epsilon}_1}{T_1} & -\frac{1}{T_1} \\ \sigma_{l2} \log_e \frac{\dot{\epsilon}_2}{T_2} & -\frac{1}{T_2} \\ \sigma_{l3} \log_e \frac{\dot{\epsilon}_3}{T_3} & -\frac{1}{T_3} \end{vmatrix}$$

$$= \frac{1}{T_1} \left( \sigma_{l3} \log_e \frac{\dot{\epsilon}_2}{T_2} - \sigma_{l2} \log_e \frac{\dot{\epsilon}_3}{T_3} \right) + \frac{1}{T_2} \left( \sigma_{l1} \log_e \frac{\dot{\epsilon}_3}{T_3} - \sigma_{l3} \log_e \frac{\dot{\epsilon}_1}{T_1} \right) + \frac{1}{T_3} \left( \sigma_{l2} \log_e \frac{\dot{\epsilon}_1}{T_1} - \sigma_{l1} \log_e \frac{\dot{\epsilon}_2}{T_2} \right) \quad (4)$$

$$\begin{aligned}
\Delta \cdot \frac{\Delta H}{R} &= \begin{vmatrix} \sigma_{l1} & 1 & \log_e \frac{\dot{\epsilon}_1}{T_1} \\ \sigma_{l2} & 1 & \log_e \frac{\dot{\epsilon}_2}{T_2} \\ \sigma_{l3} & 1 & \log_e \frac{\dot{\epsilon}_3}{T_3} \end{vmatrix} \\
&= \left( \sigma_{l3} \log_e \frac{\dot{\epsilon}_2}{T_2} - \sigma_{l2} \log_e \frac{\dot{\epsilon}_3}{T_3} \right) + \left( \sigma_{l1} \log_e \frac{\dot{\epsilon}_3}{T_3} - \sigma_{l3} \log_e \frac{\dot{\epsilon}_1}{T_1} \right) \\
&\quad + \left( \sigma_{l2} \log_e \frac{\dot{\epsilon}_1}{T_1} - \sigma_{l1} \log_e \frac{\dot{\epsilon}_2}{T_2} \right) \quad (5)
\end{aligned}$$

This approach has the advantage of computing the material constants independently and utilizing a minimum of the test conditions and data in a straightforward procedure. However, due to the manner in which much of the data was presented in the literature, a second method was found to be more advantageous for this study.

If, for any one material, there exists stress data at large strains acquired from tests at two or more constant strain rates ( $\dot{\epsilon}_1, \dot{\epsilon}_2, \dots, \dot{\epsilon}_i$ ) at the same constant temperature ( $T_1 = T_2 = \dots, T_i$ ), then Equation (A5) provides  $i - 1$  relations for the material constant  $\sigma_0$  of the following form

$$\sigma_0 = \frac{\sigma_{li} - \sigma_{l1}}{\log_e \frac{\dot{\epsilon}_i}{\dot{\epsilon}_1}} \quad (6)$$

where  $\sigma_{l1}$  is the lowest limiting stress in the group.

Knowing the value of  $\sigma_0$ , it then becomes possible to solve for  $\frac{\Delta H}{R}$ , provided values of the limiting stress exist for one strain rate ( $\dot{\epsilon}_1 = \dot{\epsilon}_2 = \dots, \dot{\epsilon}_i$ )

at two or more temperatures ( $T_1, T_2, \dots T_i$ ). Again, from Equation (A5), we have  $i - 1$  relations of the type

$$\frac{\Delta H}{R} = \frac{\frac{\sigma_{li} - \sigma_{l1}}{\sigma_0} + \log_e \frac{T_1}{T_i}}{\frac{1}{T_i} - \frac{1}{T_1}} \quad (7)$$

Finally, the value of  $s$  may be acquired from the relation

$$s = \frac{\dot{\epsilon}_i}{T_i \exp \left( \frac{\sigma_{li}}{\sigma_0} - \frac{\Delta H}{RT_i} \right)} \quad (8)$$

where  $\sigma_0$  and  $\frac{\Delta H}{R}$  are the values acquired from Equations (6) and (7), respectively and  $\dot{\epsilon}_i$ ,  $T_i$ , and  $\sigma_{li}$  are the test data from anyone of the previously used constant strain-rate tests.

Theoretically, in Equations (6) and (7), only two values of  $\sigma_l$  are required; in Equation (8), only one value is necessary. However, if more values exist, these may be used to provide an arithmetic average in the event that minor deviations for  $\sigma_0$ ,  $\frac{\Delta H}{R}$ , and  $s$  arise as a result of slight irregularities in the recorded or plotted data and to also verify the fact that the parameter is indeed constant over a range of strain rates.

Mechanical properties of various propellants are presented by Burgardt, Frazer and Britton<sup>(1)</sup> for temperatures ranging from 233° to 311°K, and loading rates of 0.05, 0.2, 2.0, 10.0 in/min. A strain gage length of 2 inches was assumed to obtain strain rates. Two propellants were selected as examples.

In deriving the parameters  $\sigma_0$ ,  $\frac{\Delta H}{R}$  and  $\log_e s$  for Propellant 1, the ultimate stress data for all loading rates at 277.4°K were used, along with the stress at 298°K, loading rate 2.0 in/min.

The ultimate stresses at other temperatures were then predicted by again using Equation (1), i. e.

$$\sigma_l = \sigma_0 \left( \frac{\Delta H}{RT} + \log_e \dot{\epsilon} - \log_e T - \log_e s \right)$$

where  $\sigma_0$ ,  $\frac{\Delta H}{R}$  and  $\log_e s$  were the previously determined constants and  $\sigma_l$  was calculated for each T and  $\dot{\epsilon}$ . The computed values compare favorably with the experimental values given for other temperatures. The ratios of experimental to calculated ultimate stresses are given in Table 1.

The same method was employed to compute the material constants of the second propellant; however, the ultimate stresses used were obtained at 298°K for all loading rates and at 277.4°F, loading rate 2.0 in/min. Ultimate stresses were then calculated and Table 2 lists the ratios of experimental to calculated stresses for the second propellant.

The calculated stresses are consistent with the experimental data for both propellants with the possible exception of data at 233°K. A reasonable estimate of the limiting temperature of Equation (1), is one-half the absolute melting temperature.

In the study of Smith's data<sup>(2)</sup> for polyisobutylene polymers the process was again utilized and the material constants were found to be

TABLE 1. RATIO OF EXPERIMENTAL TO CALCULATED  
ULTIMATE STRESSES (PROPELLANT 1)

Temperature		Rate of Loading, in/min	Ratio
<u>°K</u>	<u>°F</u>		
310.8	100	0.05	1.173
		0.2	.889
		2	1.085
		10	.923
298	77	0.05	1.173
		0.2	1.188
		2	1.000
		10	1.164
277.4	40	0.05	.977
		0.2	.980
		2	.989
		10	1.043
266.3	20	0.05	1.098
		0.2	1.135
		2	1.073
		10	1.074
255.2	0	0.05	.985
		0.2	1.119
		2	1.008
		10	1.051
233.	-40	0.5	1.196
		0.2	1.179
		2	1.178
		10	1.441

TABLE 2. RATIO OF EXPERIMENTAL TO CALCULATED  
ULTIMATE STRESSES (PROPELLANT 2)

Temperature		Rate of Loading, in/min	Ratio
<u>°K</u>	<u>°F</u>		
310.8	100	0.05	1.046
		0.2	.920
		2.0	1.044
		10.0	.921
298	77	0.05	1.027
		0.2	1.000
		2.0	1.010
		10.0	1.101
277.4	40	0.05	.846
		0.2	.930
		2.0	1.000
		10.0	1.040
266.3	20	0.05	.970
		0.2	1.046
		2.0	1.160
		10.0	1.185
255.2	0	0.05	1.081
		0.2	1.083
		2.0	1.222
		10.0	1.379
233	-40	0.05	1.259
		0.2	1.000
		2.0	1.478
		10.0	1.716

$\sigma_0 = 3.7$ ,  $\frac{\Delta H}{R} = 5200$ ,  $s = 0.37$ . The ultimate stresses were then calculated and can be found in Table 3 and Figure 1. The experimental values used to compute the parameters are noted in the table.

In addition to the ultimate strain data, experimental stress-strain data were also given. Therefore, the constants were utilized in Equation (A2) and stress-strain curves were generated for four constant strain rates at  $T = 358^\circ\text{K}$ . These curves appear in Figure 2. The difference in the high strain rate stress-strain curves is due to the initial hypotheses in which the steady-state creep condition was considered.

The material constants for a GR-S rubber were calculated from data acquired by Smith<sup>(3)</sup> at temperatures  $298^\circ$ ,  $316^\circ$ ,  $366^\circ\text{K}$  and found to be  $\sigma_0 = 15$ ,  $\frac{\Delta H}{R} = 9430$ , and  $s = 5560$ . The ultimate stresses for several strain rates at  $360.8^\circ\text{K}$  were computed and compared with Smith's experimental data in Figure 3. Smith's data are based on averages of two tests at each strain rate.

TABLE 3. ULTIMATE STRESS OF POLYISOBUTYLENE

Temperature		Strain Rate, in/in/sec	Experimental Ultimate Stress, psi	Calculated Ultimate Stress, psi
°K	°C			
241.4	-31.6	.000104	34.6	31.7
		.00104	41.6	40.9
		.0104	46.8	50.1
		.104	74.1	59.3
272	- 1.0	.000104	----	21.4
		.00104	29.3	30.6
		.0104	39.5	39.8
		.104	48.6	49.0
300	26.7	.000104	12.8*	13.8
		.00104	24.3*	23.0
		.0104	32.6	32.2
		.104	41.4	41.4
333	60	.000104	5.0*	6.6
		.00104	14.3*	15.8
		.0104	25.1*	25.0
		.104	30.4	34.2
358	85	.000104	2.8*	1.9
		.00104	9.9*	11.1
		.0104	17.7*	20.3
		.104	27.9*	29.5

\*Used in calculating parameters.

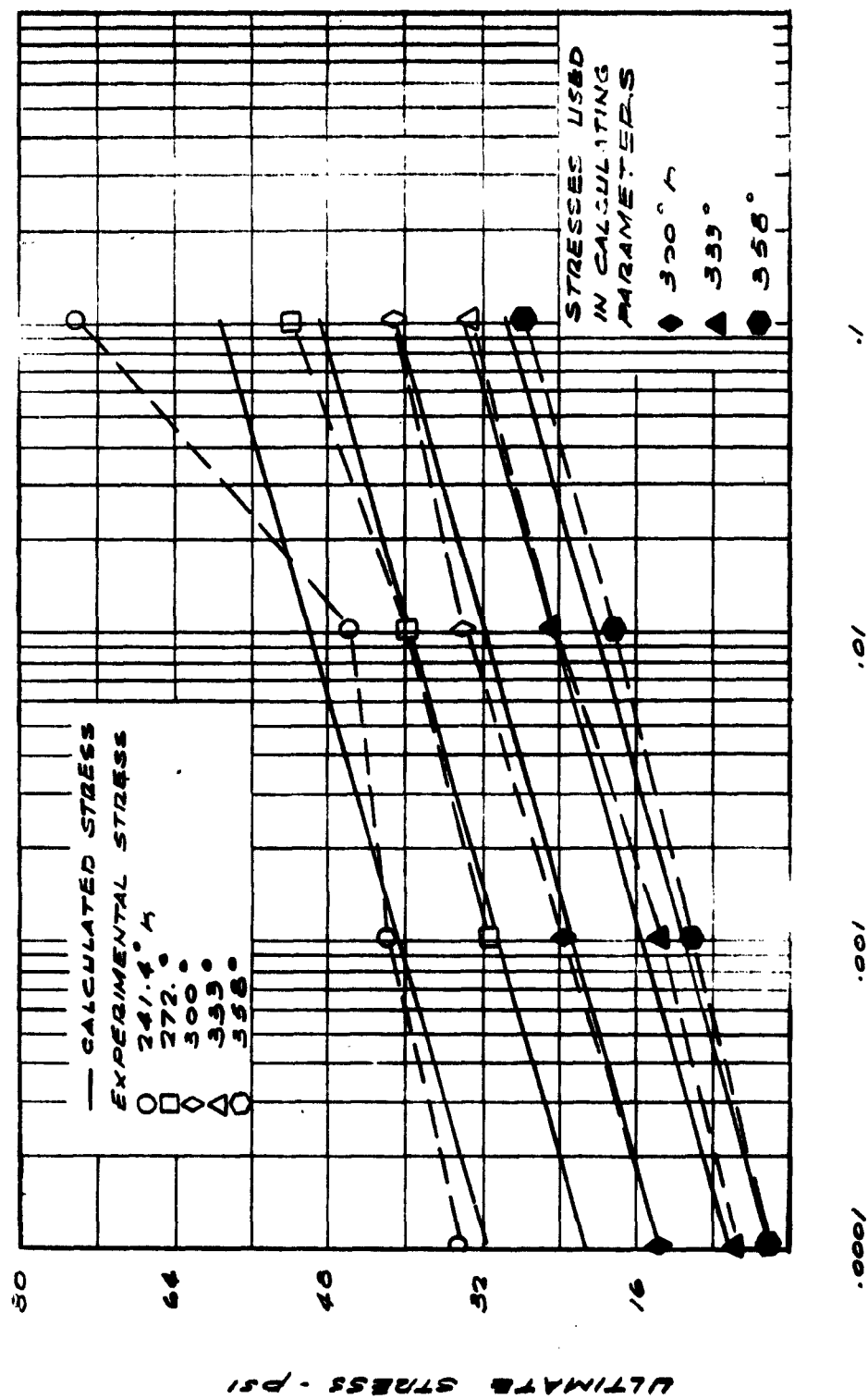


FIGURE 1. ULTIMATE STRESS OF POLYISOBUTYLENE AT VARIOUS TEMPERATURES AND STRAIN RATES

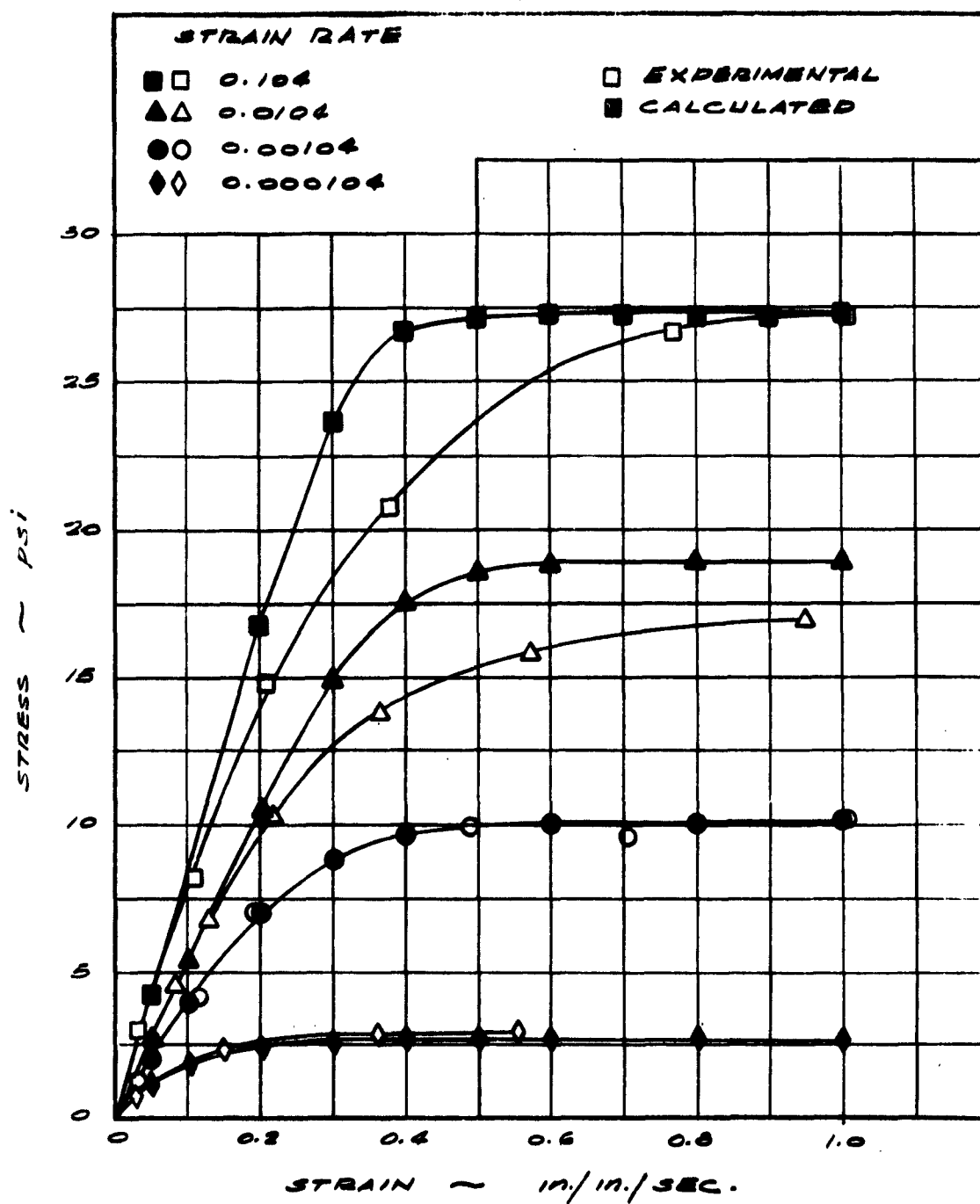


FIGURE 2. STRESS-STRAIN CURVES FOR  
POLYISOBUTYLENE AT 358°K

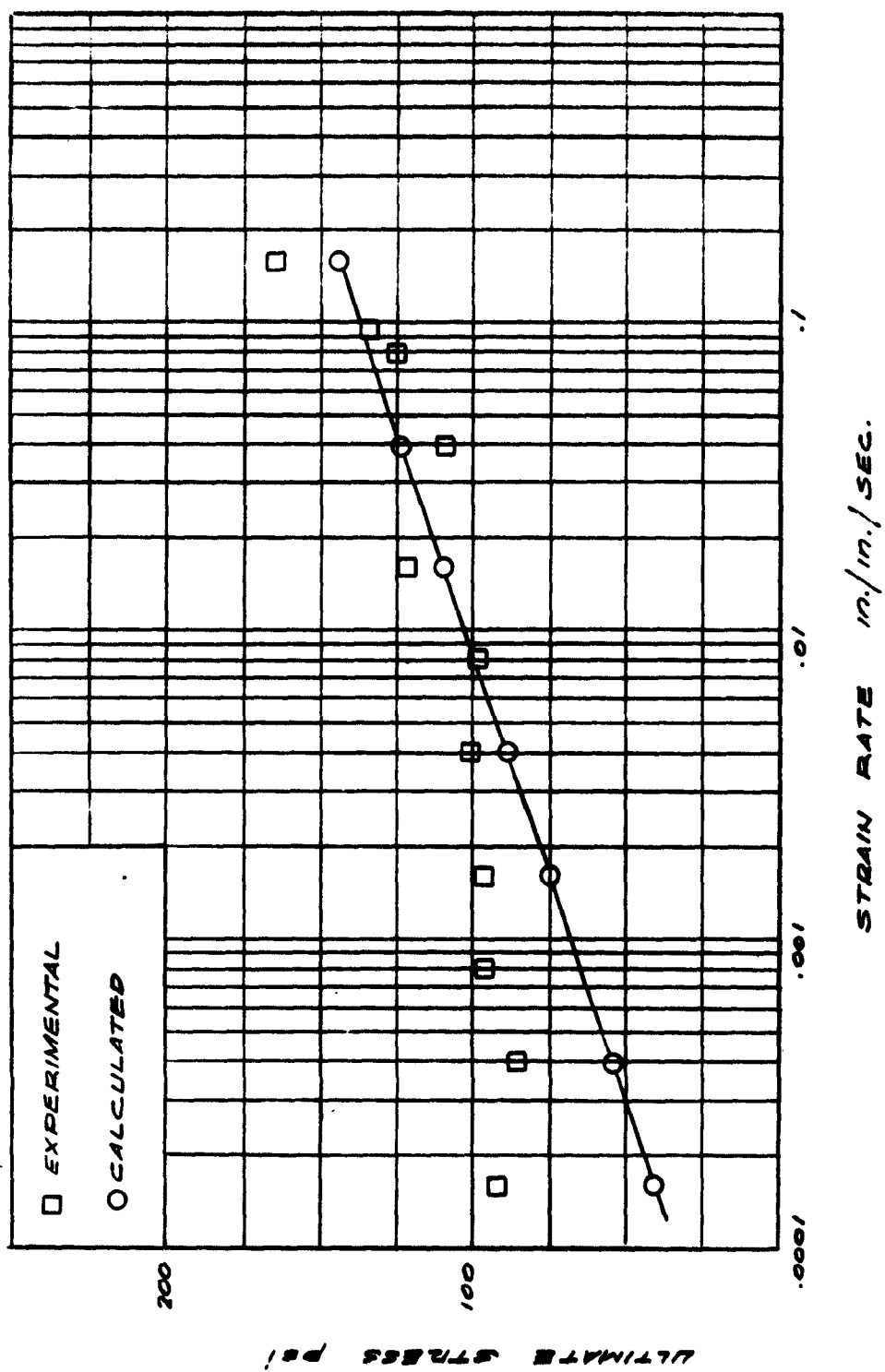


FIGURE 3. ULTIMATE STRESS OF A GR-S RUBBER AT 360. 8°K

### III. STRESS RELAXATION

Although data are limited for the test conditions where strain and temperature are constant and the stress varies with time, Tobolsky<sup>(4)</sup> has published applicable data for unfractionated polyisobutylene at 50% elongation at temperatures 303°, 323°, 343°, 373°K.

To obtain the constants  $\sigma_0$ ,  $\frac{\Delta H}{R}$ , and  $s$  we first utilize Equation (A15)

$$\tanh \frac{\sigma}{2\sigma_0} = \tanh \frac{\sigma(t_0)}{2\sigma_0} \exp \left( - \frac{t}{t'} \right)$$

Since  $t'$  is dependent upon the temperature and contains two unknown parameters, we must separately obtain  $t'_a$  and  $t'_b$ . It was necessary, therefore, to choose at  $T = T_a = 373^\circ\text{K}$  two stresses and their times of occurrence which gave a system of two equations, i. e.,

$$\begin{aligned} \tanh \frac{\sigma_1}{2\sigma_0} &= \tanh \frac{\sigma(t_0)}{2\sigma_0} \exp \left( - \frac{t_1}{t'_a} \right) \\ \tanh \frac{\sigma_2}{2\sigma_0} &= \tanh \frac{\sigma(t_0)}{2\sigma_0} \exp \left( - \frac{t_2}{t'_a} \right) \end{aligned} \quad (9)$$

where  $\sigma(t_0)$  is the known initial stress and  $t_1 = 1 \times 10^{-2}$ ,  $t_2 = 1 \times 10^{-1}$ .

Solution of this system provides  $\sigma_0$  and  $t'_a$ . Solution of Equation (9) was then repeated at  $T = T_b = 323^\circ\text{K}$  to obtain  $t'_b$ . The applied stress  $\sigma(t_0)$  must be known for each test condition. In this study,  $\sigma(t_0)$  was assumed to be very near the stress at  $t = 1 \times 10^{-3}$ ; this was proven to be a good approximation.

The known values of  $t'$  are used to generate from Equation (A16) a new system of simultaneous equations

$$\begin{aligned} t'_a &= \frac{\sigma_0 \exp\left(\frac{\Delta H}{RT_a}\right)}{2s T_a E_a} \\ t'_b &= \frac{\sigma_0 \exp\left(\frac{\Delta H}{RT_b}\right)}{2s T_b E_b} \end{aligned} \quad (10)$$

where  $E = \frac{\sigma(t_0)}{\epsilon}$  and is dependent on the respective temperatures. We found  $\sigma_0 = 4$ ,  $\frac{\Delta H}{R} = 7040$ ,  $s = 2.93 \times 10^5$ . Stress decay curves were then calculated by use of Equation (A15) and are shown in Figure 4. The experimental and calculated curves are in good agreement.

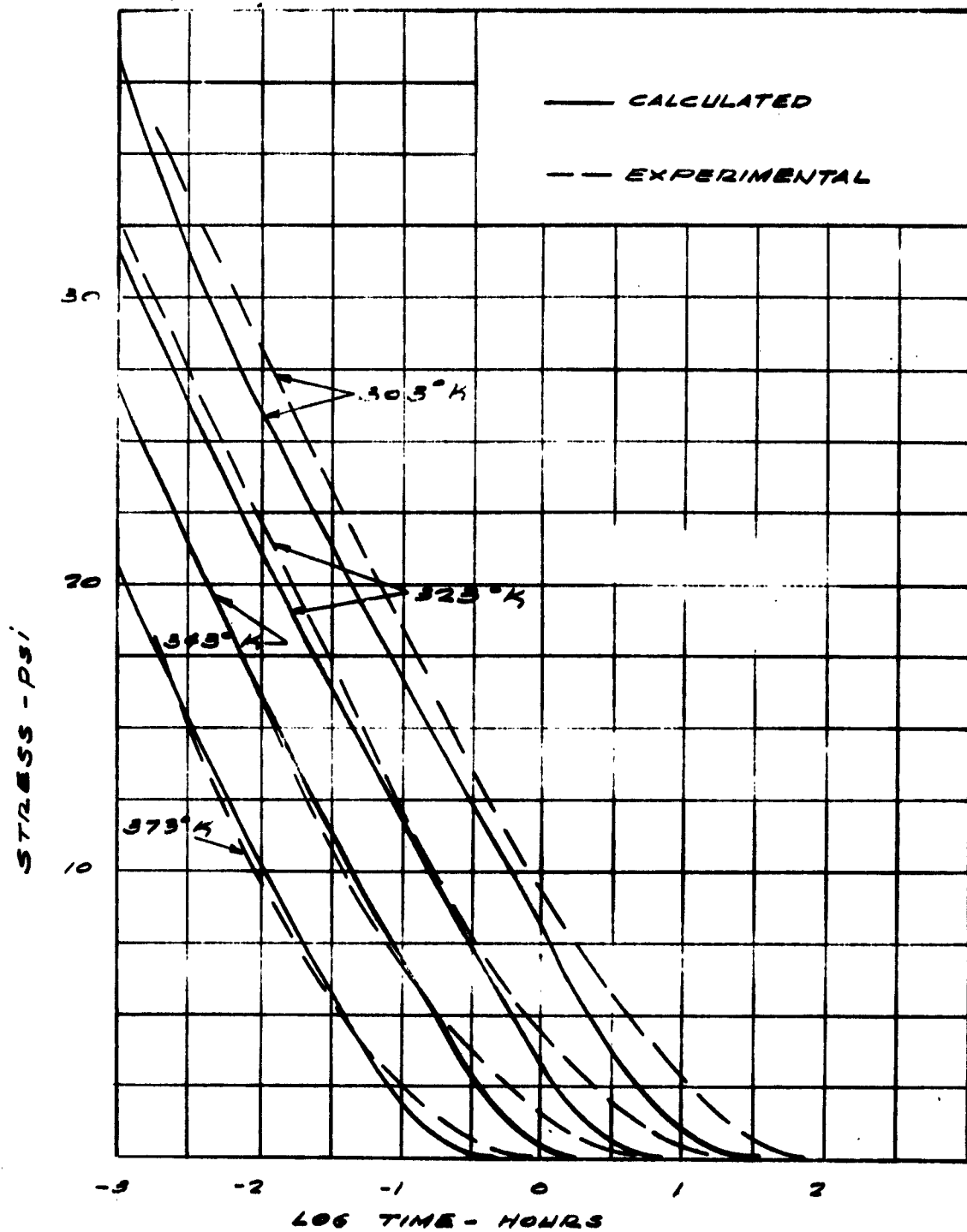


FIGURE 4. STRESS DECAY OF UNFRACTIONATED POLYISOBUTYLENE AT 50% ELONGATION

#### IV. DISCUSSION

The degree of verification of the basic, phenomenological relation for viscoelastic materials is necessarily restricted by the rather limited amount of applicable data. A brief literature study was conducted but much of the data uncovered was, for our purposes, incomplete. From the data found and presented herein, however, certain encouraging trends are seen.

Procedures for deriving the materials constants  $\sigma_0$ ,  $\frac{\Delta H}{R}$ , and  $s$  are adaptable to the amount of usable data available. With the restriction to temperatures at which the viscoelastic properties are in effect, ultimate stresses can be predicted within a close percentage. The stress-strain curves calculated correlate readily with the experimental curves, especially at slow strain rates. The calculated stress decay curves agree both in magnitude and in general form with the experimental curve, even though the actual initial load was not known and had to be estimated.

Within the limits of this study, these results are favorable, and the application of the basic phenomenological relation to other testing conditions for viscoelastic materials should be no less encouraging.

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**APPENDIX**  
**PHENOMENOLOGICAL CONCEPTS FOR**  
**VISCOELASTIC MATERIALS**

## APPENDIX

The following relations were obtained from Stowell's<sup>(A1)</sup> development and are repeated here for convenience.

$\epsilon$  = Strain

$\dot{\epsilon}$  = Strain rate

$t$  = Time

$\sigma$  = Stress

$T$  = Temperature, °K (unless otherwise indicated)

$E$  = Young's modulus

$\alpha$  = Linear expansion coefficient

$\Delta H$  = Activation energy, cal per mole

$R$  = Gas constant, taken as 2 cal per mole per °K

$s$  = constant

$\sigma_0$  = constant

## PHENOMENOLOGICAL CONCEPTS FOR VISCOELASTIC MATERIALS

### A. General

The basic phenomenological relation, which forms the foundation for mathematically describing the behavior of solid propellants, received its initial impetus from a paper by Kauzmann<sup>(A1)</sup>. In this paper, Kauzmann following a suggestion of Condon, attempted to compute the creep rate for a metal under steady stress from the viewpoint of reaction-rate theory as used by physical chemists, notably Eyring. Although the Kauzmann equation contained much truth, experimental evidence clearly indicated modifications were necessary. Three modifications plus the inclusion of terms for the effects of elasticity (always present to some extent) and thermal expansion led to the basic formulation by Stowell<sup>(A2)</sup>. This relation has since been used successfully to predict the behavior of a variety of viscoelastic material under various time and temperature conditions (for example, see Ref. A3 and A6).

The basic equation proposed for a viscoelastic material is given by:

$$\frac{d\epsilon}{dt} = \dot{\epsilon} = \frac{d}{dt} \left( \frac{\sigma}{E} \right) + a \frac{dT}{dt} + \frac{2sT \sinh \frac{\sigma}{\sigma_0}}{\exp \frac{\Delta H}{RT}} \quad (A1)$$

Essentially, the preceding phenomenological relation requires only the acquisition of the material's constants ( $\Delta H$ ,  $s$ , and  $\sigma_0$ ) from any one of

the several, more familiar quasi-static tests. Solutions to problems involving viscoelastic materials are obtained as integrals of Equation (A1) subject to the appropriate load, deformation and temperature conditions (see below). Thus, there is no recognizable "equation of state," although Equation (A1) may be appropriately called an "equation of rate of change of state."

#### B. Stress-Strain Relations

Equation (A1) yields the following stress-strain relation for constant temperatures and strain-rate conditions:

$$\sigma = \sigma_0 \log_e \frac{(1 + \beta^2)^{1/2} + (1 + \beta) \tanh M}{(1 + \beta^2)^{1/2} + (1 - \beta) \tanh M} \quad (A2)$$

where

$$\beta = \frac{\dot{\epsilon} \exp \frac{\Delta H}{RT}}{2sT} \quad M = \frac{\dot{\epsilon} E t (1 + \beta^2)^{1/2}}{2\sigma_0 \beta}$$

If  $\beta^2 \gg 1$ , then Equation (A2) reduces to

$$\sigma = \sigma_0 \log_e \frac{\beta + (1 + \beta) \tanh M}{\beta + (1 - \beta) \tanh M} \quad (\beta^2 \gg 1) \quad (A3)$$

where

$$M = \frac{E \dot{\epsilon} t}{2\sigma_0}$$

For the limiting condition where the strain and therefore  $M$  are large (so that  $\tanh M \approx 1$ ), Equation (A3) becomes

$$\sigma_f = \sigma_0 \log_e (2\beta + 1) \quad \left( \begin{array}{l} \beta^2 \gg 1 \\ \tanh M = 1 \end{array} \right) \quad (A4)$$

where  $\sigma_f$  is defined as the limiting stress associated with large strains.

Finally, if  $2\beta \gg 1$ , then

$$\sigma_f = \sigma_o \log_e \frac{\dot{\epsilon} \exp \frac{\Delta H}{RT}}{sT} \quad \left( \begin{array}{l} \epsilon \text{ is large} \\ \beta^2 \gg 1 \\ \tan M = 1 \\ 2\beta \gg 1 \end{array} \right) \quad (A5)$$

### C. Creep

For the condition where the temperature and load (stress) are constant, Equation (A1) yields the following relation for the secondary, steady creep rate:

$$\dot{\epsilon}_c = 2sT \frac{\sinh \frac{\sigma}{\sigma_o}}{\exp \frac{\Delta H}{RT}} \quad (A6)$$

If  $\sigma \gg \sigma_o$ , Equation (A6) reduces to

$$\dot{\epsilon}_c = sT \exp \left( \frac{\sigma}{\sigma_o} - \frac{\Delta H}{RT} \right) \quad (\sigma \gg \sigma_o) \quad (A7)$$

or

$$\sigma = \sigma_o \left( \log_e \dot{\epsilon}_c + \frac{\Delta H}{RT} - \log_e sT \right) \quad (A8)$$

Equation (A1), in itself, does not give information concerning the final or tertiary stage of creep nor the creep rupture. However, it is possible, with the help of some qualitative consideration, to arrive at an equation for creep rupture. For example, assuming that the strain-rate in the final stage ( $\dot{\epsilon}_f$ ) is of the form

$$\dot{\epsilon}_f = \dot{\epsilon}_c f \left( \frac{\epsilon_f}{\rho} \right) \quad (A9)$$

where  $\rho$  is a constant and taking the function to be

$$f\left(\frac{\epsilon_f}{\rho}\right) = \exp \frac{\epsilon_f}{\rho} \quad (\text{A10})$$

we have

$$\frac{d\epsilon_f}{\exp \frac{\epsilon_f}{\rho}} - \dot{\epsilon}_c dt = 0 \quad (\text{A11})$$

Integration of this relation yields

$$\frac{\dot{\epsilon}_c t_f}{\rho} = 1 - \exp\left(-\frac{\epsilon_f}{\rho}\right) \quad (\text{A12})$$

where  $t_f$  is the tertiary stage time. If  $\epsilon_f = \epsilon_r \gg \rho$  when the time to rupture is  $t_f = t_r$ , then from Equation (A12), we have

$$\rho = \dot{\epsilon}_c t_r = \text{constant} \quad (\text{A13})$$

Thus, with the values for the steady creep rate determined from Equations (A6) or (A7) and the extra constant  $\rho$ , it should be possible to calculate the creep rupture time.

#### D. Stress Relaxation

For those test conditions where the strain and temperature are maintained constant and the stress (load) changes as a function of time, we have from Equation (A1) (for  $\dot{\epsilon} = 0$ )

$$\frac{1}{E} \frac{d\sigma}{dt} + \frac{2sT}{\exp \frac{\Delta H}{RT}} \sinh \frac{\sigma}{\sigma_0} = 0 \quad (\text{A14})$$

When integrated, this yields the following expression for the rate of decay of stress with time.

$$\tanh \frac{\sigma}{2\sigma_0} = \tanh \frac{\sigma(t_0)}{2\sigma_0} \exp \left( -\frac{t}{t'} \right) \quad (\text{A15})$$

where  $\sigma(t_0)$  is the applied stress at  $t = 0$  and

$$t' = \frac{\sigma_0 \exp \frac{\Delta H}{RT}}{2sTE} \quad (\text{A16})$$

#### E. Increasing Temperatures

For the condition where the load (stress) is constant and the temperature increases at a constant rate ( $\dot{T} = \dot{T}_0$ ), Equation (A1) yields the following relation for the strain at any absolute temperature (T):

$$\epsilon = \frac{\sigma}{E} + \alpha \Delta T + \frac{2s \left( \frac{\Delta H}{R} \right)^2}{\dot{T}_0} \sinh \frac{\sigma}{\sigma_0} \left[ f \left( \frac{\Delta H}{RT} \right) \right] \quad (\text{A17})$$

where

$$f \left( \frac{\Delta H}{RT} \right) = \left[ \left( \frac{\Delta H}{RT} \right)^{-3} \exp \left( -\frac{\Delta H}{RT} \right) \right] \left[ 1 - 3 \left( \frac{\Delta H}{RT} \right)^{-1} + 12 \left( \frac{\Delta H}{RT} \right)^{-2} - \dots \right]$$

If a rod is restrained at the end and is subjected to a linearly increasing temperature, the change in thermal stress ( $\Delta\sigma$ ) may be acquired in the step-by-step solution of the following expression:

$$\Delta\sigma = E \left[ \alpha - \left( \frac{sT}{\dot{T}_0} \right) \exp \left( \frac{\sigma}{\sigma_0} - \frac{\Delta H}{RT} \right) \right] \Delta T \quad (\text{A18})$$

#### F. Periodic Stress

For the condition where the imposed load may be expressed as a periodic function:

$$\sigma = A \sin \omega t$$

When used in Equation (A1), we have

$$\dot{\epsilon} = \frac{\omega A}{E} \cos \omega t + 2sT \exp\left(-\frac{\Delta H}{RT}\right) \sinh\left(\frac{A}{\sigma_0} \sin \omega t\right) \quad (A19)$$

Equation (A19) would indicate that there is no net effect. This is due to the equal value placed on the positive and negative parts of the cycle. During the compression portion of the cycle, however, the greater part of the creep during the tension portion is not recovered. Consequently, a positive result is obtained, if the contribution of the compression portion is considered as zero. The strain per cycle is then

$$\begin{aligned} \epsilon_{pc} &= \frac{\omega A}{E} \int_0^{2\pi} \cos \omega t \, dt + \frac{2sT \exp\left(-\frac{\Delta H}{RT}\right)}{\omega} \int_0^{\pi} \sinh\left(\frac{A}{\sigma_0} \sin \omega t\right) d(\omega t) \\ &= sT\tau \exp\left(-\frac{\Delta H}{RT}\right) F\left(\frac{A}{\sigma_0}\right) \end{aligned} \quad (A20)$$

where  $\tau$  is the period of stress alternation and

$$F\left(\frac{A}{\sigma_0}\right) = \frac{4}{\pi} \left[ I_1\left(\frac{A}{\sigma_0}\right) - \frac{1}{3} I_3\left(\frac{A}{\sigma_0}\right) + \frac{1}{5} I_5\left(\frac{A}{\sigma_0}\right) - \dots \right]$$

$I_n$  being the modified Bessel function.

Assuming rupture occurs in the same manner as when the metal is under constant stress:

$$\dot{\epsilon}_c t_r = \epsilon_r$$

and the mean value of  $\dot{\epsilon}_c$  over a cycle is

$$\dot{\epsilon}_c = \frac{\epsilon_{pc}}{\tau}$$

and also

$$t_r = N_r \tau$$

where  $N_r$  is the number of cycles required for rupture. Therefore,

$$\epsilon_{ctr} = N_r \epsilon_{pc} = \epsilon_r$$

or

$$\epsilon_r = N_r sT \exp \left( - \frac{\Delta H}{RT} \right) F \left( \frac{A}{\sigma_0} \right) \quad (A21)$$

This equation contains only known quantities with the exception of  $N_r$  and, theoretically, could be solved for  $N_r$ . Thus, for the first time we have a fatigue equation with no empirical factors.

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